Jenny and the Magic Bean

Unable to find a pair of solar-filter glasses to watch a solar eclipse, Jenny decides to craft a pinhole camera from a box of her favourite cereal, Super-Ohs!

The first step in constructing a pinhole camera is to cut a rectangle from a sheet of white copy paper such that it has the same dimensions as the base of the cereal box. The piece is then placed at the bottom of the box to be used as a projection screen.

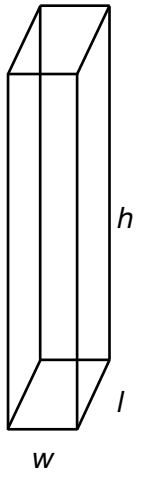
Jenny likes a challenge. So, rather than tracing the box's base on the copy paper as a guide to cut the rectangle, she decides to measure the base. Unfortunately, she broke her ruler the other night trying to swat a fly!

Suddenly, a Magic Bean materializes before Jenny. The Magic Bean says:

> "Jenny: do not despair! I have three secrets about the cereal box that will help you!"
>
> "Secret #1: The box's volume, $V(x)$, is $10x^3 + 65x^2 + 100x + 45$ cm^3.
>
> "Secret #2: The box's height is five times the width of the base.
>
> "Secret #3: The perimeter of the base is 50 cm.

The Magic Bean disappears. Jenny, a math whiz, understands now what to do.

Jenny correctly figures out the dimensions of the base, cuts the copy paper and watches the solar eclipse safely in all its pinhole-projected glory.



What are the dimensions of the cut copy paper Jenny needs for her pinhole camera?

Solution

The first step is to factor the volume to get the dimensions of the box in terms of x. Start by doing some greatest common factoring:

$$10x^3 + 65x^2 + 100x + 45 = 5(2x^3 + 13x^2 + 20x + 9)$$

Now, concentrating on the polynomial $2x^3 + 13x^2 + 20x + 9$, let's use Factor Theorem to see if we can find a zero. We'll see if we can avoid writing out all the possible rational roots by trying ± 1 first.

Let $P(x) = 2x^3 + 13x^2 + 20x + 9$. If $x = 1$, then

$P(1) = 2(1)^3 + 13(1)^2 + 20(1) + 9 = 44$ #No good

$P(-1) = 2(-1)^3 + 13(-1)^2 + 20(-1) + 9 = 0$ #That works!

So since -1 is a zero of $P(x)$, then $x + 1$ is a factor of $P(x)$.

Next, let's divide $P(x)$ by $x + 1$ to find another factor. You can use long division here, if you prefer, but synthetic division is faster and more concise.

$$\begin{array}{r|cccc} & 2 & 13 & 20 & 9 \\ -1 & & -2 & -11 & -9 \\ \hline & 2 & 11 & 9 & 0 \end{array}$$

Thus, we can say that $P(x) = (x + 1)(2x^2 + 11x + 9)$, and continue as follows:

$$\begin{aligned} P(x) &= (x+1)(2x^2 + 11x + 9) \\ &= (x+1)(2x^2 + 2x + 9x + 9) \\ &= (x+1)(2x(x+1) + 9(x+1)) \\ &= (x+1)(x+1)(2x+9) \end{aligned}$$

So,

$$\begin{aligned} V(x) &= 10x^3 + 65x^2 + 100x + 45 \\ &= 5(2x^3 + 13x^2 + 20x + 9) \\ &= 5(x+1)(x+1)(2x+9) \\ &= (5x+5)(x+1)(2x+9) \end{aligned}$$

Thus, knowing that the volume of the cereal box is $l \times w \times h$ and Secret #2, we can deduce each dimension: $h = 5x + 5$, $w = x + 1$, and $l = 2x + 9$.

Given that the perimeter is 50 cm, then

$$\begin{aligned} 2l + 2w &= \text{Perimeter} \\ 2(2x + 9) + 2(x + 1) &= 50 \\ 4x + 18 + 2x + 2 &= 50 \\ 6x + 20 &= 50 \\ 6x &= 30 \\ x &= 5 \end{aligned}$$

So, $h = 5(5) + 5 = 30$ cm, $w = 5 + 1 = 6$ cm, and $l = 2(5) + 9 = 19$ cm.

Therefore, Jenny must cut a 6 cm × 19 cm piece of copy paper for her pinhole camera.